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THESE

présentée

à l'UNIVERSITE DE PARIS - SUD

CENTRE D'ORSAY

pour obtenir le titre de Docteur 3e Cycle

SPECIALITE : MATHEMATIQUE

par B.S. KIRANAGI

Sujet de la Thèse : Lie Algebra Fibre Bundles

Soutenue le 17 juin 1976 devant la Commission d'examen

MM. A. Douady Président

M. Demazure } Examinateurs

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ACKNOWLEDGEMENTS

I express my deep sense of gratitude to Professor P. Cartier for his valuable guidance, suggestions and encouragement during this study. I am highly thankful to Professors R.L.E. Schwarzenberger and T.S. Bhanumusty for introducing to this field and for their constant encouragement.

I am sincerely thankful to Professors A. Douady and M. Demazure for accepting to be the members of the jury. My sincere thanks are due to Professor Poèto and the staff of the departmental library, Orsay for providing me the facilities (and I am also thankful to the administrative and library staff of I.H.E.S. for their valuable help during the course of this study).

I am very much thankful to Mme Cartier for the pains she has taken during the preparation of this thesis. I am also very much thankful to Mme Parvan for her excellent typing of this manuscript. The help rendered by friends is highly appreciated.

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INTRODUCTION

The notion of a Lie algebra bundle has been introduced by A. Douady and M. Lazard [4]. A Lie algebra bundle ξ over a topological space X is a vector bundle ξ over X, with a morphism $\theta : \xi \oplus \xi \longrightarrow \xi$ which induces a Lie-algebra structure on each fibre ξ_x. In this thesis we develop a theory of real Lie algebra bundles over a topological space, though much of the elementary theory is similar for vector bundles and Lie algebra bundles in which all fibres are isomorphic as Lie algebras.

In chapter 1, we give a few basic definitions for Lie algebra fibre bundles.

Chapter 2. We recall the definition of a Lie algebra bundle and give some examples. We prove our main theorem 2.7 that if all the fibres ξ_x of a Lie algebra bundle ξ over a connected space X are isomorphic as Lie algebras, then ξ is locally a trivial Lie algebra bundle (a Lie algebra bundle which is isomorphic to the product Lie algebra bundle). But if the base space and total space are manifolds and all the morphisms are C^r, $r \geq 1$, then the above result is found in [5, theorem 1 p. 373]. However their techniques do not generalize to topological spaces and continuous maps. We also prove "the Existence theorem" for Lie algebra bundles in which all fibres are

isomorphic as Lie algebras adopting the same techniques in case of vector bundles.

Chapter 3. We study some properties of semi-simple Lie algebra bundles (A Lie algebra bundle in which all fibres are semi-simple). Mainly we prove the "splitting theorem" and "Krull-Schmidt theorem" for semi-simple Lie algebra bundles in which all fibres are isomorphic as Lie algebras.

Chapter 4. We prove the full analogue of "Swan's theorem" [9 theorem 2, p. 268] by giving a characterization of which Lie rings occur as the spaces of sections of a Lie algebra bundles.

0. NOTATIONS AND PRELIMINARIES

Let $\mathbb{R}$ denote the field of real numbers. A fibre bundle $\xi = (E(\xi),p,X)$ over a topological space X (the base space) consists of a space $E(\xi)$ (the total space), a continuous map $p : E(\xi) \longrightarrow X$ (the projection) which is onto, and, on each fibre $\xi_x = E_x(\xi) = p^{-1}(x)$, the structure of a finite dimensional vector space over $\mathbb{R}$ with the topology compatible with the topology on ξ_x induced from $E(\xi)$.

A fibre bundle $\eta = (E(\eta), p_\eta, X')$ is a fibre subbundle of $\xi = (E(\xi),p_\xi,X)$ provided $E(\eta)$ is a subspace of $E(\xi)$, X' is a subspace of X and $p_\eta = p_\xi|_{E'(\eta)}$ and each fibre η_x is a subspace of ξ_x.

A fibre bundle ξ is called a vector bundle if for each $x \in X$, there is a neighborhood U of x in X and a homeomorphism $\varphi : p^{-1}(U) \longrightarrow U \times V$ such that the restriction $\varphi_x = \varphi|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow V$ is $\mathbb{R}$-linear. Here V is a fixed vector space, called a standard fibre of ξ and (U,φ) is called a local chart at x.

A vector bundle η is called a vector subbundle of ξ if it is a fibre subbundle of ξ.

A cross section S of $\xi = (E(\xi),p,X)$ is a continuous map $: X \longrightarrow E(\xi)$ such that $pS = I_X$. It follows from the definition of a vector bundle that for any $x \in X$, there is a neighborhood U of x and cross sections $\{S_i\}_{i=1}^{i}$ of ξ over U such that $\{S_i(x)\}$ forms a basis for each ξ_x, $x \in U$ such a set $\{S_i\}$ is called a local base at x.

The Whitney sum of two fibre bundles $\xi_1 = (E(\xi_1), p_{\xi_1}, X)$ and $\xi_2 = (E(\xi_2), p_{\xi_2}, X)$ is $\xi_1 \oplus \xi_2 = (E(\xi_1) \oplus E(\xi_2), q, X)$ where $E(\xi_1) \oplus E(\xi_2)$ is the subspace of all $(e_1, e_2) \in E(\xi_1) \times E(\xi_2)$ with $p_{\xi_1}(e_1) = p_{\xi_2}(e_2)$ and $q(e_1, e_2) = p_{\xi_1}(e_1) = p_{\xi_2}(e_2)$. The fibre $q^{-1}(x)$ of $\xi_1 \oplus \xi_2$ over $x \in X$ is $p_{\xi_1}^{-1}(x) \times p_{\xi_2}^{-1}(x)$.

Finally we observe that the Whitney sum of two vector-bundles ξ_1 and ξ_2 is again a vector bundle $\xi_1 \oplus \xi_2$. If $\varphi_1 : U \times V_1 \longrightarrow p_{\xi_1}^{-1}(U)$ and $\varphi_2 : U \times V_2 \longrightarrow p_{\xi_2}^{-1}(U)$ are local charts of ξ_1 and ξ_2 respectively. Then $\varphi_1 \oplus \varphi_2 : U \times (V_1 \oplus V_2) \longrightarrow q^{-1}(U)$ is a local chart of $\xi_1 \oplus \xi_2$ where $\varphi_1 \oplus \varphi_2 (x, v_1, v_2) = (\varphi_{1,x}(v_1), \varphi_{2,x}(v_2))$, $v_1 \in V_1$, $v_2 \in V_2$ and $x \in X$. The standard fibre of $\xi_1 \oplus \xi_2$ is $V_1 \oplus V_2$ where V_1 and V_2 are standard fibres of ξ_1 and ξ_2.

All our bundles are real unless otherwise specified and we denote them by Greek letters ξ, η, ν ... . Generally we denote the total space $E(\xi)$ of a bundle ξ by ξ itself, the fibres by ξ_x and the projection map by p_ξ.

1. LIE ALGEBRA FIBRE BUNDLE

In this chapter we give a few definitions which we need throughout our work.

DEFINITION 1.1. A Lie (algebra) fibre bundle is a fibre bundle ξ together with a morphism $\theta : \xi \oplus \xi \longrightarrow \xi$ which induces a Lie algebra structure on each fibre ξ_x.

Here after we call the morphism $\theta : \xi \oplus \xi \longrightarrow \xi$ the Lie structure of ξ. We denote a Lie fibre bundle (ξ,θ) by ξ itself since we are considering only one Lie structure θ_ξ for ξ.

Example 1.2. Any fibre bundle with each fibre having an abelian Lie algebra structure is a Lie fibre bundle and is called an abelian Lie fibre bundle.

DEFINITION of morphism 1.3. Let $\xi = (E(\xi),p_\xi,X)$, $\xi' = (E(\xi'),p_{\xi'},X')$ be two Lie fibre bundles. A Lie fibre bundle morphism

$$(\varphi,f) : \xi \longrightarrow \xi'$$

is a pair of continuous maps

$$\varphi : E(\xi) \longrightarrow E(\xi'), \quad f : X \longrightarrow X'$$

such that the following diagram is commutative

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\varphi} & E(\xi') \\ {\scriptstyle p_\xi}\downarrow & & \downarrow{\scriptstyle p_{\xi'}} \\ X & \xrightarrow[f]{} & X' \end{array} \qquad \text{i.e. } f\, p_\xi = p_{\xi'}\varphi,$$

and for each $x \in X$, the map $\varphi_x : \xi_x \longrightarrow \xi_{f(x)}$ is a Lie algebra homomorphism.

We say that a morphism is an isomorphism if φ is bijective and φ^{-1}, f^{-1} are continuous maps. If two Lie fibre bundles have the same base space X then we take f = Identity : $X \longrightarrow X$.

Example 1.4. Let $\xi = (X \times L, p, X)$ be the product vector bundle with $p : X \times L \longrightarrow X$ given by $p(x,\ell) = x$.

Let L be a Lie algebra with the Lie bracket $[,]$. Then

$$\theta : \xi \oplus \xi \longrightarrow \xi \quad \text{given by}$$

$$\theta(x,\ell,\ell') = (x,[\ell,\ell'])$$

is continuous and induces on each fibre $L_x = L$ the given Lie algebra structure of L. This bundle is called the Lie product bundle and any Lie fibre bundle which is isomorphic to the Lie product bundle is called a trivial Lie bundle.

DEFINITION 1.5. A Lie fibre bundle (η,θ_η) is called a Lie fibre subbundle of a Lie fibre bundle (ξ,θ_ξ) if it is a fibre subbundle of ξ and each fibre η_x is a Lie subalgebra of ξ_x.

Restriction of bundles is defined as follows :

DEFINITION 1.6. Let (ξ,θ_ξ) be a Lie fibre bundle and let Y be a

subspace of X. Then the restriction of ξ to Y, denoted $\xi_{|Y}$, is the Lie fibre bundle $\xi_{|Y} = (E(\xi_{|Y}), p_{\xi_{|Y}}, Y)$ where $E(\xi_{|Y}) = p_\xi^{-1}(Y)$, $p_{\xi|_Y} = p_{|E(\xi_{|Y})}$ and $\theta_{\xi|_Y} = \theta_{\xi|_{E(\xi_{|Y})} \oplus E(\xi_{|_Y})}$.

The Whitney sum of two Lie fibre bundles (ξ,θ_ξ), (η,θ_η) is again a Lie fibre bundle $\xi \oplus \eta$ with the Lie structure

$$\theta_\xi \oplus \theta_\eta : (\xi \oplus \eta) \oplus (\xi \oplus \eta) = (\xi \oplus \xi) \oplus (\eta \oplus \eta) \longrightarrow \xi \oplus \eta$$

defined by $\theta_\xi \oplus \theta_\eta((e_1,e_2) \oplus (e_1',e_2')) = \theta_\xi(e_1,e_1') \oplus \theta_\eta(e_2,e_2')$ for all $e_1, e_2 \in \xi$, $e_1', e_2' \in \eta$. Obviously $\theta_\xi \oplus \theta_\eta$ induces a Lie algebra structure on each fibre $\xi_x \times \eta_x$.

DEFINITION 1.7. The Whitney sum of two Lie fibre bundles (ξ,θ_ξ) and (η,θ_η) is the fibre product of the under lying fibre bundles with the above given Lie structure $\theta_\xi \oplus \theta_\eta$.

2. LIE ALGEBRA BUNDLES.

Our basic field is always the set $\mathbb{R}$ of real numbers.

In this chapter we recall the definition of a Lie algebra bundle and construct new examples. Mainly we prove that if all fibres of ξ are isomorphic as Lie algebras then ξ is locally a trivial Lie algebra bundle (isomorphic to product bundle). So that one can develop Lie algebra bundle theory analoguously to vector bundle theory.

DEFINITION 2.1. *A vector bundle is called a Lie (algebra) bundle if it is a Lie fibre bundle.*

Example 2.2. Let L be a non abelian Lie algebra with basis $\{e_i\}_{i=1}^{n}$. Then $(I \times L, p, I)$ is a trivial vector bundle where $I = [0,1]$.

Consider a map $\theta : I \times (L \oplus L) \longrightarrow I \times L$

$$(t,(\ell,\ell')) \longrightarrow (t,\theta_t(\ell,\ell'))$$

where $\theta_t : (\ell,\ell') = t.[\ell,\ell']$, $\ell, \ell' \in L$, $t \in I$.

Then θ is continuous and obviously it induces a Lie algebra structure on each fibre L_t. Since let β be the set of all $\mathbb{R}$-bilinear maps : $L \times L \to L$, then β is a vector space over $\mathbb{R}$ and therefore the scalar product $\mathbb{R} \times \beta \to \beta$ is continuous. Then $I \to I.[\ ]$ is continuous.
$t \to t.[\ ]$

Thus $(I \times L, p, I)$ is a Lie bundle with the above defined Lie algebra structure θ.

Before giving some more examples we prove our main theorem needed for later use.

DEFINITION 2.3. A Lie bundle η is called a Lie subbundle of a Lie bundle ξ if η is a vector subbundle of ξ and if each fibre η_x is a Lie subalgebra of ξ_x. If each fibre η_x is an ideal of ξ_x, then η is called an ideal subbundle of ξ.

DEFINITION 2.4. A morphism of Lie bundles is a morphism of the underlying fibre bundles.

THEOREM 2.5. Let $\varphi : \xi \to \eta$ be a Lie bundle morphism over a basic space X. Then φ is an isomorphism if and only if $\varphi_x : \xi_x \to \eta_x$ is a Lie algebra isomorphism for each $x \in X$.

Proof. Suppose φ is an isomorphism. Then by the definition of morphism it follows that each $\varphi_x : \xi_x \to \eta_x$ is a Lie algebra isomorphism.

Conversely define a morphism $\psi : \eta \to \xi$ such that for each $x \in X$, $\psi_x : \eta_x \to \xi_x$ is the inverse of φ_x. Then ψ is continuous [6, theorem 2.5, p. 25] and obviously ψ_x is a Lie algebra homomorphism.

Thus φ is an isomorphism. Q.E.D.

We will now prove our main theorem that if all the fibres ξ_x of a Lie bundle ξ are isomorphic as Lie algebras then it is locally a trivial Lie

bundle. To prove this we need to prove that every real orbit of a real point p is open in the real part of the corresponding complex orbit.

Let $V_{\mathbb{C}} = \mathbb{C}^n$ be a vector space over the field $\mathbb{C}$ of complex numbers. Then $B_{\mathbb{C}}$ = the set of all $\mathbb{C}$-bilinear maps : $V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ is a complex algebraic variety being a vector space over $\mathbb{C}$. The complex algebraic group $G_{\mathbb{C}} = GL(n,\mathbb{C})$ acts morphically on the complex algebraic variety $B_{\mathbb{C}}$ as follows :

$$G_{\mathbb{C}} \times B_{\mathbb{C}} \longrightarrow B_{\mathbb{C}}$$

$$(g,\alpha) \longmapsto g.\alpha$$

where $(g.\alpha)(u,v) = g\ \alpha(g^{-1}u, g^{-1}v)$, for $g \in G_{\mathbb{C}}$, $\alpha \in B_{\mathbb{C}}$, $u,v \in V$.

Otherwise stated

$$(g.\alpha)^k_{ij} = \sum_{\ell,m,n} g^k_n\, g^\ell_i\, g^m_j\, \alpha^n_{\ell m}$$

where $g = (g^\ell_k)$, $g^{-1} = (g^k_\ell)$ and $\alpha = (\alpha^n_{\ell m})$. The subset $G_{\mathbb{C}} \circ \varphi_o = \{g \circ \varphi_o : g \in G_{\mathbb{C}}\}$ is called the orbit of the point $\varphi_o \in B_{\mathbb{C}}$ and the closed subgroup $G_{\varphi_o} = \{g \mid g \circ \varphi_o = \varphi_o\}$ of $G_{\mathbb{C}}$ is called the stability subgroup of $\varphi_o \in B_{\mathbb{C}}$.

Let $B_{\mathbb{R}}$ be the set of all elements of $B_{\mathbb{C}}$ which map $V_{\mathbb{R}} \times V_{\mathbb{R}} \to V_{\mathbb{R}}$ where $V_{\mathbb{R}} = \mathbb{R}^n \subset \mathbb{C}^n$. Then $B_{\mathbb{R}} \subset B_{\mathbb{C}}$ and $G_{\mathbb{R}} = GL(n,\mathbb{R})$ acts morphically on $B_{\mathbb{R}}$. Here we will prove that the real orbit $G_{\mathbb{R}} \circ \varphi_o$ of a real point $\varphi_o \in B_{\mathbb{R}}$ is open in the real part of the complex orbit $G_{\mathbb{C}} \circ \varphi_o$.

LEMMA 2.6. The real orbit of a real point is open in the real part of its complex orbit.

Proof. The complex algebraic group $G_{\mathbb{C}}$ acts morphically on the complex alge-

braic variety $B_{\mathbb{C}}$. Then every orbit is a smooth variety which is open in its closure [1, closed orbit lemma 1.8, p. 98]. Hence the orbit $X_{\mathbb{C}} = G_{\mathbb{C}} \circ \varphi_o$ of a real point $\varphi_o \in B_{\mathbb{R}}$ is a smooth manifold. The morphism $\psi : G_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ defined by

$$\psi : g \longmapsto g.\varphi_o$$

is surjective morphism of manifolds.

Let $\mathcal{G}_{\mathbb{C}}$ be the tangent space of $G_{\mathbb{C}}$ at the identity element and $T_{\varphi_o}(X_{\mathbb{C}})$ be the tangent space of $X_{\mathbb{C}}$ at φ_o . Therefore the differential $d\psi$ of ψ, say $d\psi : \mathcal{G}_{\mathbb{C}} \longrightarrow T_{\varphi_o}(X_{\mathbb{C}})$ is su jective, where $d\psi(X)f = X(f \circ \psi)$, for $X \in \mathcal{G}_{\mathbb{C}}$ and $f \in \mathcal{A}_{\varphi_o}(X) =$ the set of differentiable functions around φ_o.

Consider now $\mathcal{G}_{\mathbb{R}}$ and $T_{\varphi_o}(X_{\mathbb{C}})^{\mathbb{R}}$, the Lie algebra of $G_{\mathbb{R}}$ and the intersection $T_{\varphi_o}(X_{\mathbb{C}}) \cap T_{\varphi_o}(B_{\mathbb{R}})$ respectively.

Then observe that $d\psi$ induces a surjective morphism from : $\mathcal{G}_{\mathbb{R}}$ onto $T_{\varphi_o}(X_{\mathbb{C}})^{\mathbb{R}}$. Indeed $d\psi$ is surjective, and compatible with the natural complex conjugations in $\mathcal{G}_{\mathbb{C}}$ (with fixed point set $\mathcal{G}_{\mathbb{R}}$) and in $T_{\varphi_o}(X_{\mathbb{C}})$ (with fixed point set $T_{\varphi_o}(X_{\mathbb{C}})^{\mathbb{R}} = T_{\varphi_o}(X_{\mathbb{C}}) \cap T_{\varphi_o}(B_{\mathbb{R}})$.

Now let $X_{\mathbb{R}} = X_{\mathbb{C}} \cap B_{\mathbb{R}}$, consist of the real points in the complex orbit $X_{\mathbb{C}}$. As there exists a neighborhood U of the identity in $G_{\mathbb{R}}$ consisting of one-parameter subgroups of $G_{\mathbb{R}}$ and their orbits of φ_o fill up a neighborhood of φ_o in $X_{\mathbb{R}}$, i.e. $V \circ \varphi_o$ is an open neighborhood of $X_{\mathbb{R}}$ and $\varphi_o \in U \circ \varphi_o \subset G_{\mathbb{R}} \circ \varphi_o$. For any $g \circ \varphi_o \in G_{\mathbb{R}} \circ \varphi_o$, $g \circ U \circ \varphi_o$ is open in $X_{\mathbb{R}}$ and $g \circ \varphi_o \in g\, V \circ \varphi_o \subset G_{\mathbb{R}} \circ \varphi_o$. Therefore $G_{\mathbb{R}} \circ \varphi_o$ is open in $X_{\mathbb{R}} = (G_{\mathbb{C}} \circ \varphi_o) \cap B_{\mathbb{R}}$.

Hereafter by isomorphism of fibres of Lie bundle we mean Lie algebra isomorphism, unless otherwise specified.

Now we prove a theorem, for a Lie bundle in which all fibres are isomorphic, which helps to extend most of the properties of vector bundles to the case of Lie bundles.

THEOREM 2.7. *Let* $\xi = (E,p,X)$ *be a Lie bundle over a base space* X. *Assume that the fibres in* ξ *are mutually isomorphic. Then for all* $x \in X$, *there exists a neighborhood* U *of* x_o, *a fixed Lie algebra* L *and a homeomorphism* $\sigma : U \times L \longrightarrow p^{-1}(U)$ *such that the restriction* $\sigma_x : x \times L \longrightarrow p^{-1}(x)$ *is a Lie algebra homomorphism for each* $x \in U$.

Proof. Suppose ξ is a Lie bundle in which all fibres are mutually isomorphic.

By assumption there is a continuous map $\theta : \xi \oplus \xi \longrightarrow \xi$ such that

$$\theta_x : \xi_x \times \xi_x \longrightarrow \xi_x$$

$$\theta_x(u,v) = \varphi_x(u,v)$$

induces a Lie algebra structure φ_x on each ξ_x.

Suppose $\xi^r = U_r \times V$, where U_r is an open subset of X and V is a fixed vector space with basis $\{v_i\}_{i=1}^n$. Hence the commutation diagram :

$$\begin{array}{ccc} \theta^r : U_r \times (V \oplus V) & \longrightarrow & U_r \times V \\ \downarrow & \swarrow & \\ U_r & & \end{array}$$

One has $$\theta^r(x,(v_i,v_j)) = (x,\varphi_x^r(v_i,v_j))$$

i.e. $$\varphi_x^r(v_i,v_j) = \sum_\ell C^\ell_{r,ij}(x)\, v_\ell .$$

Since θ_r is continuous, $C^\ell_{r,ij}$ is continuous too. Therefore the mapping $x \longmapsto \varphi_x^r = (C^\ell_{r,ij}(x))$ from U_r to B_* is continuous where B_* is the set of all Lie brackets : $V \times V \longrightarrow V$ with the topology induced from the set B_0 of all bilinear maps.

Moreover B is an algebraic variety being a vector space. We know that the algebraic group $G = GL(n,\mathbb{R})$ acts morphically on B with the following action $G \times B \longrightarrow B$

$$(g,\alpha) \longmapsto g.\alpha$$

where $g.\alpha(u,v) = g\,\alpha(g^{-1}u,g^{-1}v)$ for $u, v \in V$. Let φ_o be the Lie bracket of V at some point $0 \in U_r$ and $\tilde{B} = G \circ \varphi_o$ is the orbit of φ_o in B. We now construct a continuous map from U_r to $\tilde{B}$.

Also by assumption for all $x, y \in X$, the Lie algebras ξ_x and ξ_y are isomorphic. Therefore $\theta = \theta_{x,y} : V = \xi_x^r \longrightarrow \xi_y^r = V$ is a Lie algebra isomorphism,

i.e. $$\theta\, \varphi_x(v_i,v_j) = \varphi_y(\theta\, v_i,\ \theta\, v_j),\quad \varphi_x,\varphi_y \in B_* .$$

Hence we have $\varphi_x = \theta_{o,x}\varphi_o$ and we have the required map

$$X \longrightarrow \tilde{B} = G \circ \varphi_o,$$

continuous with respect to the induced topology from B.

Now by making use of Aren's theorem [8, Theorem p. 65] we will prove that $\tilde{B}$ is homeomorphic to $G/_H$ where H is the stability group of φ_o.

By the lemma 2.6, the real orbit $\tilde{B} = G \circ \varphi_o$ if a real point $\varphi_o \in B$ is

open in the real part $B_{\mathbb{R}} \cap \tilde{B}_{\mathbb{C}}$ of its complex orbit $G_{\mathbb{C}} \circ \varphi_o = \tilde{B}_{\mathbb{C}}$. Since $\tilde{B}_{\mathbb{C}}$ is locally closed for the Zariski topology [1, closed orbit Lemma 1.8 p. 98] the same is true in the ordinary topology. Therefore $\tilde{B}$ is open in $\bar{\tilde{B}}_{\mathbb{C}} \cap B_{\mathbb{R}}$. Therefore $\tilde{B} = \tilde{B} \cap \bar{\tilde{B}}$ is open in $\bar{\tilde{B}}$. That is $\tilde{B}$ is open in $\bar{\tilde{B}}$. Therefore $\tilde{B}$ is a locally closed subset of B.

Since B is a locally compact Hausdorff space with respect to the usual topology, $\tilde{B}$ is a locally compact Hausdorff subspace of B, being a locally closed subset of B [2, proposition 5, p. 120].

Since $G = GL(n,\mathbb{R})$ is a locally compact Hausdorff space with respect to the usual topology. Therefore G contains an open subgroup G' such that G'/G_o is compact, where G_o is the identity component of G.

As G has only two components $G_o = \{s \in G | \det s > 0\}$ and $G_1 = \{s \in G | \det s < 0\}$, we can take $G' = G_o$ and $G/G' = \{1,-1\}$ is countable.

Thus $\tilde{B}$ and G satisfies all the required conditions of the Aren's theorem. Therefore $\tilde{B}$ is homeomorphic to G/H where H is the stability subgroup of φ_o.

So we have a continuous map : $U^r \longrightarrow \tilde{B} \longrightarrow G/H$. Since G is a Lie group and H is a closed subgroup of G, therefore H has a local cross section therefore $G \longrightarrow G/H$ is a locally trivial principal bundle. Hence by shrinking if necessary the open sets U^r, we can assume that there exists for each r a continuous map $\theta^r : U_r \longrightarrow GL(n,\mathbb{R})$ such that for each x in U_r, one gets

$$\varphi_x^r = \theta^r(x) \circ \varphi_o .$$

Otherwise stated, $\theta^r(x)$ is an isomorphism of the fixed Lie algebra ξ_o onto the Lie algebra ξ_x.

Thus $T : p^{-1}(U_r) = U_r \times V \longrightarrow U_r \times V$

$$(x,v) \longrightarrow (x, \theta^r(x)^{-1}(v))$$

is a homeomorphism and obviously restricted to each fibre $p^{-1}(x) \longrightarrow V$ is a Lie algebra homomorphism.

The theorem is then proved.

We can now prove a converse to theorem 2.7. Assume that $\xi = (E,p,X)$ is given where E and X are connected topological spaces and p is a continuous map. Assume that $p(E) = X$ and that each fibre $E_x = p^{-1}(x)$ has been given the structure of a Lie algebra. Assume moreover that each point in X belongs to an open set U for which there exists a Lie algebra L and a homeomorphism $T : U \times L \longrightarrow p^{-1}(U)$ such that

$$T_x : L \longrightarrow p^{-1}(x)$$

is a Lie algebra isomorphism for every x in U. We claim that ξ is a Lie bundle in which the fibres are mutually isomorphic.

But the following diagram is commutative

$$\begin{array}{ccc} U \times L \times L & \xrightarrow{T \oplus T} & p^{-1}(U) \oplus p^{-1}(U) \\ [\ ]_1 \downarrow & & \downarrow [\ ] \\ U \times L & \xrightarrow{T} & p^{-1}(U) \end{array}$$

i.e. $[\ ] . T \oplus T = T \circ [\ ]_1$

where $[\ ]_1(x,\ell,\ell') = (x, [\ell,\ell']_1)$ and $T \oplus T\ (x,(\ell,\ell')) = (T(x,\ell),T(x,\ell'))$. Therefore $[\ ] \circ T \oplus T = T \circ [\ ]_1$ is continuous being the composition of two continuous functions T, $[\ ]_1$. But $T \oplus T$ is homeomorphism. Therefore $[\ ]$ is continuous on $p^{-1}(U) \oplus p^{-1}(U) \longrightarrow p^{-1}(U)$ and the coordinate neighborhoods $\{U\}$ cover X. Therefore $\theta : \xi \oplus \xi \longrightarrow \xi$ defined by

$$(v,v') \longrightarrow [v,v']_x$$

is continuous. Thus ξ is a Lie bundle.

Thus the converse is proved.

COROLLARY 2.8. *Suppose X is a Hausdorff space and ξ is a Lie bundle over X in which the fibres are mutually isomorphic. Then for every $x \in X$, there exists a neighborhood U of x in X and a Hausdorff Lie group bundle [4] over U, whose Lie algebra bundle is isomorphic to $\xi_{|U}$.*

Proof. By the above theorem 2.7, we know that for every $x \in X$, there exists a neighborhood U of x and morphism $T : p^{-1}(U) \longrightarrow U \times L$ is a Lie bundle isomorphism. Therefore the corresponding Lie group bundle $U \times G$ over U of $U \times L$ is Hausdorff, where G is a simply connected Lie group.

Thus the corollary follows.

Remark 2.9.

The above theorem 2.7 is not true if all fibres are not isomorphic as Lie algebras.

For, in the example 2.2, the fibre ξ_o at $0 \in I$ is abelian and for

all $t \neq 0 \in I$, the fibres ξ_t are non abelian. Therefore $0 \in I$, there does not exist a neighborhood U of 0 and a homeomorphism

$$p^{-1}(U) \longrightarrow U \times L$$

such that $p^{-1}(x) \longrightarrow x \times L$ is a Lie algebra homomorphism.

Example 2.10. Let $\xi = (E,p,X)$ be a vector bundle over an arbitrary base space X with a standard fibre V of dimension n where $V_x = p^{-1}(x)$.

Notice that $$E = \bigcup_{x \in X} p^{-1}(x) = \bigcup_{x \in X} V_x.$$

Now consider the bundle $\bar{\xi} = (\bar{E},\bar{p},X)$ where $\bar{p}^{-1}(x) = \mathrm{End}(V_x)$ for each $x \in X$; then $\bar{p}^{-1}(x)$ is a Lie algebra with $[f,g] = f \circ g - g \circ f$, where $g, f \in \mathrm{End}(V_x)$. Then $(\bar{E},\bar{p},X)$ is a Lie bundle where $\bar{p} : \bar{E} \longrightarrow X$ defined by

$$\bar{p}(f) = x \quad \text{if} \quad f \in \mathrm{End}(V_x).$$

For, we know that for any $x \in X$, there exists a neighborhood U of x such that

$$\varphi_U : U \times V \longrightarrow p^{-1}(U)$$

is a vector bundle isomorphism into $\xi = (E,p,X)$.

Consider now $\overline{\varphi_U} : U \times \mathrm{End}(V_x) \longrightarrow \bar{p}^{-1}(U)$ with $\overline{\varphi_U}(x,f) = \varphi_{U_x} f \varphi_{U_x}^{-1}$. Then $\overline{\varphi_U}$ is a morphism and φ_{U_x} is bijective for all $x \in U$, and

$$\overline{\varphi_{U_x}} : \mathrm{End}(V) \longrightarrow \bar{p}^{-1}(x)$$

is a Lie algebra homomorphism, i.e.

$$\overline{\varphi_{U_x}}[f,g] = [\overline{\varphi_{U_x}} f, \overline{\varphi_{U_x}} g], \quad f, g \in \mathrm{End}(V,V).$$

For
$$[\overline{\varphi_{U_x}} f, \overline{\varphi_{U_x}} g] = [\varphi_{U_x} f \varphi_{U_x}^{-1}, \varphi_{U_x} g \varphi_{U_x}^{-1}]$$
$$= \varphi_{U_x} f \varphi_{U_x}^{-1} \varphi_{U_x} g \varphi_{U_x}^{-1} - \varphi_{U_x} g \varphi_{U_x}^{-1} f \varphi_{U_x}^{-1}$$
$$= \varphi_{U_x} f g \varphi_{U_x}^{-1} - \varphi_{U_x} g f \varphi_{U_x}^{-1}$$
$$= \varphi_{U_x} [f,g] \varphi_{U_x}^{-1}$$
$$= \overline{\varphi_{U_x}} [f,g].$$

Then by the theorem φ_{U_x} is an isomorphism.

Thus by the converse to theorem 2.7 $\bar{\xi}$ is a Lie bundle.

We will now prove the "Existence theorem" of Lie bundles in which all fibres are isomorphic. With the help of this theorem we will show the existence of some more examples.

Let G be a topological group and X a space. By a system of co-ordinate transformations in X with values in G is meant an indexed covering $\{U_i\}_{i\in I}$ of X by open subsets and a collection of continuous maps

$$g_{ji} : U_i \cap U_j \longrightarrow G, \quad i,j \in J,$$

such that $g_{kj}(x)\, g_{ji}(x) = g_{ki}(x), \quad x \in U_i \cap U_j \cap U_k$.

EXISTENCE THEOREM 2.11. If G is a topological transformation group of a Lie algebra L such that for each fixed $g \in G$, the map $\ell \longmapsto g.\ell$ from L into L is a Lie algebra homomorphism and $\{U_i\}$, $\{g_{ij}\}$ is a system of co-ordinate transformations in the space X, then there exists a Lie bundle E with base space X, fibre L, group G and the co-ordinate transformations

$\{g_{ij}\}$.

Proof. Let J be the indexing set for the open covering $\{U_i\}$ with discrete topology.

Let $T = \{(x,\ell,i) \mid x \in U_i,\ \ell \in L,\ i \in J\}$.

Then T is a topological space and is the disjoint union of open sets $U_i \times L \times i$.

Define in T an equivalence relation

$$(x,\ell,i) \sim (x',\ell',k)$$

if $$x = x',\ g_{ki}(x).\ell = \ell'.$$

This relation is a proper equivalence relation.

Define E to be the set of equivalence classes for this relation in T.

Let $q : T \longrightarrow E$ by $q(x,\ell,i) = \{(x,\ell,i)\}$.

A set B in E is called open if and only if $q^{-1}(B)$ is an open set of T.

Then E is a topological space and q is continuous.

Define $p : E \longrightarrow X$ by $p\{(x,\ell,i)\} = x$.

Then p is continuous. Define $\varphi_i : U_i \times L \longrightarrow p^{-1}(U_i)$ by $\varphi_i(x,\ell) = q(x,\ell,i)$.

Then φ_i is a homeomorphism and

$$\varphi_{ix}^{-1} \circ (\varphi_{jx}(\ell)) = g_{ij}(x).\ell,$$

where $\varphi_{kx}(\ell) = \varphi_k(x,\ell)$.

Now we shall define the Lie algebra structure on each fibre $L_x = p^{-1}(x)$ so that φ_{ix} becomes a Lie algebra homomorphism.

Let $b_1,\ b_2 \in p^{-1}(x),\ x \in U_i,\ \alpha, \beta \in \mathbb{R}$.

1) Define $\alpha b_1 + \beta b_1 = \varphi_{ix}(\alpha\ \varphi_{ix}^{-1}(b_1) + \beta\ \varphi_{ix}^{-1}(b_2))$.

Clearly this definition is independent of the co-ordinate function φ_i.

2) $[b_1,b_2] = \varphi_{ix}([\varphi_{ix}^{-1}(b_1),\ \varphi_{ix}^{-1}(b_2)])$.

This definition is also independent of the co-ordinate function φ_i.

For, $\varphi_{jx}([\varphi_{jx}^{-1}(b_1),\ \varphi_{jx}^{-1}(b_2)]) = \varphi_{ix}\ g_{ij}(x)\ ([\varphi_{jx}^{-1}(b),\ \varphi_{jx}^{-1}(b_2)])$.

Since $\varphi_{ij}(x) = \varphi_{ix}^{-1}\ \varphi_{jx}$ and by assumption we know that $g_{ij}(x) : L \longrightarrow L$ is a Lie algebra homomorphism. Therefore $[b_1,b_2] = \varphi_{ix}([\varphi_{ix}^{-1}(b_1),\ \varphi_{ix}^{-1}(b_2)])$. Thus $p^{-1}(x)$ is a Lie algebra with the above definition and $\varphi_{ix} : L \longrightarrow p^{-1}(x)$ is a Lie algebra homomorphism.

For $\ell_1,\ell_2 \in L$ then $\varphi_{ix}([\ell_1,\ell_2]) = [\varphi_{ix}(\ell_1),\ \varphi_{ix}(\ell_2)]$. Since for $\ell_1,\ \ell_2 \in L$, there exist $b_1,\ b_2 \in p^{-1}(x)$ such that $\ell_1 = \varphi_{ix}^{-1}(b_1),\ \ell_2 = \varphi_{ix}^{-1}(b_2)$. Therefore $\varphi_{ix}[\ell_1,\ell_2] = \varphi_{ix}[\varphi_{ix}^{-1}(b_1),\ \varphi_{ix}^{-1}(b_2)]$

$= [b_1,b_2]$ by definition

therefore $\varphi_{ix}[\ell_1,\ell_2] = [\varphi_{ix}(\ell_1),\ \varphi_{ix}(\ell_2)]$.

Thus we have constructed a vector bundle in which all fibres are Lie algebras and for all $x \in X$, there exists a neighborhood U of x and a U-isomorphism $\varphi : p^{-1}(U) \longrightarrow U \times L$ such that $\varphi_x : p^{-1}(x) \longrightarrow L$ is a Lie algebra homomorphism. Then by the converse to theorem 2.7 it is a Lie bundle.

We will now construct an example with the help of the above "Existence theorem".

Example 2.12. Let $\mathfrak{Gl}(n,\mathbb{R})$ be the Lie algebra of all $n \times n$ matrices of

trace zero and $SO(n)$ be the special real orthogonal group.

We define an action of $SO(n)$ on $\mathfrak{Sl}(n,\mathbb{R})$ as follows

$SO(n) \times \mathfrak{Sl}(n,\mathbb{R}) \longrightarrow \mathfrak{Sl}(n,\mathbb{R})$ by

$$\varphi(u,x) = u\, x\, u^{-1},\ u \in SO(n),\ x \in \mathfrak{Sl}(n,\mathbb{R}).$$

Then $SO(n)$ is a topological transformation group of $\mathfrak{Sl}(n,\mathbb{R})$ and for each fixed $u \in SO(n)$ the map : $\mathfrak{Sl}(n,\mathbb{R}) \longrightarrow \mathfrak{Sl}(n,\mathbb{R})$

$$x \longrightarrow u\, x\, u^{-1}$$

is a Lie algebra automorphism, continuous and its inverse is also continuous.

Since $SO(n)$ is a Lie group, therefore there are locally trivial principal bundles $(E,p,X,\ SO(n))$ having $SO(n)$ as structure group. So the existence of co-ordinate functions $g_{ij} : U_i \cap U_j \longrightarrow SO(n)$ is assured, where $\{U_i\}$ is an open covering of X. Then by the above "Existence theorem" there exists a Lie bundle $\xi = (E,p,X,\mathfrak{Sl}(n,\mathbb{R}),\ SO(n))$ with the base space X, fibre $\mathfrak{Sl}(n,\mathbb{R})$, group $SO(n)$ and the co-ordinate transformations $\{g_{ij}\}$.

3. SEMI-SIMPLE LIE BUNDLES.

In this chapter we shall study some properties of semi-simple Lie bundles in which all fibres are isomorphic. Our main aim is to prove the splitting theorem and the Krull-Schmidt theorem.

Throughout this section a Lie bundle, subbundle, ideal bundle means a Lie bundle, subbundle, ideal-bundle in which all fibres are isomorphic as Lie algebras (hence locally trivial by theorem 2.7) unless otherwise specified and arbitrary bundles means bundles in which fibres need not be isomorphic.

DEFINITION 3.1. A semi-simple Lie bundle is a vector-bundle ξ in which the morphism $\theta : \xi \oplus \xi \to \xi$ induces a semi-simple Lie algebra structure on each fibre ξ_x.

ξ is called a simple Lie bundle if θ induces on each fibre ξ_x a simple Lie algebra structure.

THEOREM 3.2. Let $(X \times V, p, X)$ be a trivial vector bundle and $(X \times L, p_L, X)$ be a trivial semi-simple Lie bundle. Suppose $\varphi : X \times V \to X \times L$ be a vector bundle monomorphism such that for each $x \in X$, $\varphi_x(V)$ is an ideal in L. Then there exists a finite open partition $\bigcup_i X_i = X$ such that $\varphi_x(V) = \varphi_y(V)$ for all $x, y \in X_i$.

Proof. The morphism $\varphi : X \times V \longrightarrow X \times L$ defines a continuous map $\hat{\varphi} : X \to End(V,L)$. Let $\{I_i\}_{i=1}^{\ell}$ be the set of all ideals in L whose dimension equals that of V and let $X_i = \{x \in X | \varphi_x(V) = I_i\}$. Then $X = \bigcup_i X_i$. It is more sufficient to show that X_i is open.

Let X_i' be the set of linear transformation from V into I_i, then X_i' is a closed subset of $End(V,L)$ being a linear subspace of the vector space $End(V,L)$. Since $\hat{\varphi}$ is continuous. Therefore $\hat{\varphi}^{-1}(X_i')$ is a closed subset of X. Then $\hat{\varphi}^{-1}(X_i')$ is open being the complement of the closed subset $\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \hat{\varphi}^{-1}(X_i')$ in X. For any x in $\hat{\varphi}^{-1}(X_i')$, one has $\varphi_x(V) \subset I_i$ hence $\varphi_x(V) = I_i$ since $\dim V = \dim I_i$ and φ_x is a monomorphism. It follows $X_i = \hat{\varphi}^{-1}(X_i')$ hence X_i is open as claimed. For x,y in X_i, we have $\varphi_x(V) = \varphi_y(V)$.

Thus the theorem is proved.

Remark 3.3.

If X is connected, then for all $x,y \in X$

$$\varphi_x(V) = \varphi_y(V).$$

DEFINITION 3.4. Let ξ be a Lie bundle over X. A killing metric on ξ is a continuous function $K : \xi \oplus \xi \longrightarrow \mathbb{R}$ such that, for all $x \in X$, $K_{|\xi_x \times \xi_x}$ is a killing form on ξ_x (i.e. bilinear, symmetric, invariant form).

A killing metric is said to be non-degenerate if $K_{|\xi_x \times \xi_x}$ is a non-degenerate killing form on ξ_x for all $x \in X$.

THEOREM 3.5. Every semi-simple Lie bundle admits a non-degenrate killing metric.

Proof. Define $K : \xi \oplus \xi \longrightarrow \mathbb{R}$ such that, for all $x \in X$, $K_{|\xi_x \times \xi_x}$ is the canonical killing form on ξ_x i.e. $K(u,v) = Tr(ad\ u \circ ad\ v)$. Then K is continuous. For, we know that for all $x \in X$ there exists a neighborhood U of x and a morphism

$$\varphi : U \times L \longrightarrow p^{-1}(U)$$

which is a Lie bundle isomorphism and the following diagram is commutative

$$\begin{array}{ccc} U \times L \oplus L & \xrightarrow{\varphi \oplus \varphi} & p^{-1}(U) \oplus p^{-1}(U) \\ \bar{K} \downarrow & \swarrow & \\ \mathbb{R} & & \end{array}$$

i.e. $K \circ (\varphi \oplus \varphi) = \bar{K}$.

Since $K(\ell_1, \ell_2) = K(T(\ell_1), T(\ell_2))$ if $T : L \longrightarrow p^{-1}(x)$ is any isomorphism, then K is continuous since $\bar{K}$ is continuous being a bilinear form and $\varphi \oplus \varphi$ is a Lie bundle isomorphism.

Thus every semi-simple Lie bundle admits a non-degenerate killing form.

THEOREM 3.6. Let ξ be a (locally trivial) semi-simple Lie bundle over a base space X and η be a vector subbundle of ξ over X such that each η_x is an ideal in ξ_x. Then η is a Lie subbundle of ξ.

Proof. The point under question is to prove that η is a locally trivial Lie bundle.

For all $x_o \in X$, there exists a neighborhood U of x_o and U-morphism : $p_\xi^{-1}(U) \longrightarrow U \times L$ such that the restriction : $p_\xi^{-1}(x) \longrightarrow x \times L$ be a Lie algebra isomorphism : this assertion follows from theorem 2.7. By assumption η is a vector subbundle of ξ. Therefore we can choose a suitable neighborhood U of x_o such that

$$U \times V \xrightarrow{\theta} p_\eta^{-1}(U) \xhookrightarrow{i} p_\xi^{-1}(U) \xrightarrow{\varphi} U \times L$$

where θ is a vector bundle isomorphism i is an inclusion and φ is a Lie bundle isomorphism. This implies that $\varphi \circ i \circ \theta : U \times V \longrightarrow U \times L$ is continuous and for each $x \in X$,

$$\begin{aligned}(\varphi \circ i \circ \theta)_x(V) &= (\varphi \circ i)_x(\theta_x(V)) \\ &= (\varphi_x \circ i_x)(p_\eta^{-1}(x)) \\ &= \varphi_x(p_\eta^{-1}(x)) \text{ is an ideal in } L.\end{aligned}$$

Then by theorem 3.2 there exists an open partition $\bigcup_i U_i = U$ such that for all x in U_i $(\varphi \circ i \circ \theta)_x(V)$ depends only on i, not on x. Therefore $(\varphi \circ i)_x\, p_\eta^{-1}(x)) = (\varphi \circ i)_y(p_\eta^{-1}(y))$, therefore say, for all $x \in X$, $(\varphi \circ i)_x(p_\eta^{-1}(x)) = I$. Then the morphism $\varphi \circ i : p_\eta^{-1}(U_i) \longrightarrow U_i \times I$ is an Lie bundle isomorphism.

Thus η is a (locally trivial) Lie subbundle of ξ.

COROLLARY 3.7. *Let ξ be a semi-simple Lie bundle and η be a Lie bundle. If $\varphi : \xi \longrightarrow \eta$ is an epimorphism. Then* ker φ *is a Lie subbundle of* ξ.

Proof. Since ker φ is a vector subbundle of ξ [6, corollary 8.4, p. 35]

and obviously each $\ker \varphi_x$ is an ideal in ξ_x. Therefore by the above theorem $\ker \varphi$ is a Lie subbundle of ξ.

COROLLARY 3.8. *Let ξ be a Lie bundle and η be a semi-simple Lie bundle. If $\varphi : \xi \to \eta$ is a monomorphism such that each $\varphi_x(\xi)_x$ is an ideal in η_x. Then $\varphi(\xi)$ is a Lie subbundle of η and the cokernel, $\eta/\varphi(\xi)$ is also a Lie bundle over X.*

Proof. Since $\operatorname{Im} \varphi$ is a vector subbundle of η [6, corollary 8.3, p. 35] and by assumption each $\varphi_x(\xi_x)$ is an ideal in η_x, then by the above theorem $\operatorname{Im} \varphi$ is a Lie subbundle of η.

Now we will prove that $\eta/\varphi(\xi)$ is a Lie bundle over X. Indeed $\varphi(\xi)$ is a Lie subbundle of η and η can be written as the Whitney sum of two Lie fibre bundles, namely $\varphi(\xi)$ and $\varphi(\xi)^{\perp}$, i.e. $\eta = \varphi(\xi) \oplus \varphi(\xi)^{\perp}$, where $\varphi(\xi)^{\perp} = \{x' \in \xi \mid K(x,x') = 0, \quad x \in \eta\}$. Therefore the morphism

$$\theta : \eta = \varphi(\xi) \oplus \varphi(\xi)^{\perp} \longrightarrow \varphi(\xi)$$

defined by $\quad x \oplus x' \longrightarrow x$

is an epimorphism. Therefore $\ker \theta = \varphi(\xi)^{\perp}$ is a Lie subbundle of η.

Since the map $\varphi(\xi)^{\perp} \to \eta/\varphi(\xi)$ given by $x \mapsto \bar{x}$ is a Lie isomorphism, therefore $\eta/\varphi(\xi)$ is a Lie bundle over X.

LEMMA 3.9. *If $0 \to X \times L_1 \xrightarrow{\Phi} X \times L \xrightarrow{\Psi} X \times L_2 \to 0$ is an exact sequence of trivial (product) semi-simple Lie bundles over a connected base space X, then there exists an isomorphism $X \times L \xrightarrow{\sim} (X \times L_1) \oplus (X \times L_2)$*

Proof. Indeed, by theorem 3.2, $\varphi_x(L_1) = I$, for all $x \in X$, where I is some ideal in L. Consider $I^{\perp}$, the orthogonal complement of I with respect to the killing form K. Then the morphism

$$\theta = \psi_{|X\times I^{\perp}} : X \times I^{\perp} \longrightarrow X \times L_2$$

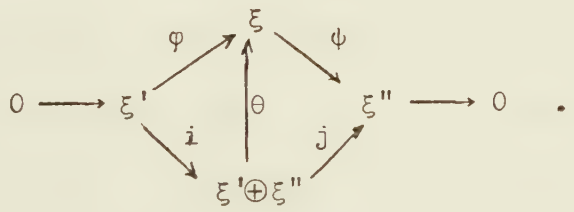
is a Lie bundle isomorphism.

Thus $X \times L \tilde{=} (X \times L_1) \oplus (X \times L_2)$.

Remark.

If X is not connected there exists a partition $X = \bigcup_i X_i$ of X into open subsets such that the conlusion of lemma 3.9 be valid for each X_i in place of X.

SPLITTING THEOREM 3.10. Let $0 \longrightarrow \xi' \xrightarrow{\varphi} \xi \xrightarrow{\psi} \xi'' \longrightarrow 0$ be an exact sequence of (locally trivial) semi-simple Lie bundles over a base space X and let K be the killing metric on ξ. Then there exists an isomorphism $\theta : \xi' \oplus \xi'' \longrightarrow \xi$ splitting the above exact sequence in the sense that the following diagram is commutative



The isomorphism i is inclusion into the first factor, and j is projection onto the second factor.

Proof. Let $\eta = \varphi(\xi') = \ker \psi$. Then by corollary 3.7, η is a Lie subbundle

of ξ. Let $\eta^{\perp}$ be the subset consisting of the elements v' in ξ such that $K(v,v') = 0$ for all $x \in \eta \subset \xi$ with $p_{\xi}(v) = p_{\xi}(v')$. Then we claim that $\eta^{\perp}$ is a Lie subbundle of ξ and isomorphic to ξ''. Obviously $\psi' = \psi_{|\eta^{\perp}} : \eta^{\perp} \to \xi''$ is bijective, continuous and, restricted to each fibre, $\psi'_x : \eta^{\perp}_x \to \xi''_x$ is a Lie algebra homomorphism.

Since, by the above lemma 3.9, ψ'^{-1} is locally continuous, therefore ψ'^{-1} is continuous.

Thus $\eta^{\perp}$ is a Lie subbundle of ξ and isomorphic to ξ''.

Thus the theorem is proved.

THEOREM 3.11. Let ξ be a (locally trivial) semi-simple Lie bundle over a base space X. Let ξ_1, ξ_2 be ideal subbundles of ξ. Then the intersection of the ideal bundles ξ_1 and ξ_2, denoted $\xi_1 \cap \xi_2$ is an ideal subbundle of ξ over X.

Proof. Consider $\xi_1 \cap \xi_2 = \bigcup_{x \in X} (\xi_{1x} \cap \xi_{2x})$ and define

$$p_3 : \xi_1 \cap \xi_2 \to X$$

by $$p_3(e) = p_1(e) = p_2(e) \quad \text{for } e \in \xi_1 \cap \xi_2.$$

It is now sufficient to prove that $\xi_1 \cap \xi_2$ is a locally trivial Lie bundle over X.

Let L_1, L_2, L be the standard Lie fibres for ξ_1, ξ_2 and of ξ repectively. For $x_0 \in X$ we can choose a suitable neighborhood U of x_0 such that the following diagrams are commutative,

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\psi} & U \times L, \\ \downarrow & \swarrow & \\ U & & \end{array} \qquad \begin{array}{ccc} U \times L_1 & \xrightarrow{\varphi_1} & p_1^{-1}(U) \\ \downarrow & \swarrow & \\ U & & \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} U \times L_2 & \xrightarrow{\varphi_2} & p_2^{-1}(U) \\ \downarrow & \swarrow & \\ U & & \end{array}$$

where ψ, φ_1, φ_2 are Lie isomorphisms. Then the morphism $\psi \circ i \circ \varphi_1 : U \times L_1 \xrightarrow{\varphi_1} p_1^{-1}(U) \xrightarrow{i} p^{-1}(U) \xrightarrow{\psi} U \times L$ implies that for all $x \in U$, $(\psi \circ i \circ \varphi_1)_x(L_1)$ is an ideal in L. Therefore by theorem 3.2 there exists an open partition $\{U_i\}_{i \in J} = U$ such that $\psi \circ i \circ \varphi_1$ has a constant image on each U_i that is $(\psi \circ i \circ \varphi_1)_y(L_1) = I_1$, for all $y \in U_i$ where I_1 is some ideal in L. Therefore $(\psi \circ i \circ \varphi_1)(U_i \times L_1) = U_i \times I_1$. Then $\psi_1 = \psi^{-1}|_{U_i' \times I_1} : U_i \times I_1 \longrightarrow p_1^{-1}(U_i)$ is a Lie isomorphism. By the similar argument we have $\psi_2 = \psi^{-1}|_{U_i' \times I_2} : U_i' \times I_2 \longrightarrow p_2^{-1}(U_i')$ is a Lie isomorphism, where $\{U_i'\}_{i \in J}$ is an open partition of U with respect to the morphism $\psi \circ i \circ \varphi_2$. As $x \in U = \bigcup_{i \in J} U_i = \bigcup_{i \in J'} U_i'$ implies that $x \in U_i$ for some $i \in J$ and $x \in U_j'$ for somme $j \in J'$. Therefore $U_i \cap U_j' = U_k$ is a neighborhood of x. Then the morphism $\psi_3 = \psi_1|_{U_k \times (I_1 \cap I_2)} = \psi_2|_{U_k \times (I_1 \cap I_2)} : U_k \times (I_1 \cap I_2) \longrightarrow p_3^{-1}(U)$ is a Lie isomorphism. Thus $\xi_1 \cap \xi_2$ is locally a trivial Lie bundle. Therefore $\xi_1 \cap \xi_2$ is an ideal bundle of ξ.

Remark 3.12.

The above theorem 3.11 need not be true in case of non-semi-simple Lie bundles. For example, let $\xi = ([0,1] \times \mathbb{R}^2, p_\xi, [0,1])$ and $\eta = ([0,1] \times \mathbb{R}, p_\eta, [0,1])$ be two trivial Lie bundles where $\mathbb{R}$ and $\mathbb{R}^2$ are considered as commutative Lie algebras.

Now we define two subbundles ξ_1, ξ_2 of ξ as follows.

Let $\phi_i : \eta \longrightarrow \xi$ defined by

$$\left.\begin{aligned} \phi_1(t,r) &= (t,\ r\ e^{2\pi it}) \\ \text{and} \qquad \phi_2(t,r) &= (t,\ r\ e^{-2\pi it}) \end{aligned}\right\} \text{ for } t \in [0,1],\ r \in \mathbb{R},$$

where $\mathbb{R}^2$ is identified with $\mathbb{C}$. Then ϕ_1, ϕ_2 are monomorphisms. Therefore $\xi_1 = \phi_1(\eta)$, $\xi_2 = \phi_2(\eta)$ are two subbundles of ξ. But $\xi_1 \cap \xi_2$ is not a subbundle of ξ as the fibres of $\xi_1 \cap \xi_2$ over $t = \frac{n}{4}$, $n \in \mathbb{N}$ has dimension 1 and over $t \neq \frac{n}{4}$, $n \in \mathbb{N}$ the fibres are zero dimensional. Thus $\xi_1 \cap \xi_2$ is not even a vector subbundle of ξ.

KRULL-SCHMIDT-THEOREM 3.13. *If ξ is a (locally trivial) semi-simple Lie bundle over X and*

$$\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n = \eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k$$

where ξ_i and η_j are simple ideal bundles, then $n = k$ and the η_j's coincides with ξ_i's (except for the order).

Proof. Consider $\xi_1 \cap \eta_j$, $j = 1, \ldots, k$. By theorem 3.11 we know that $\xi_1 \cap \eta_j$ is an ideal bundle in ξ_1 and η_j. Hence if $\xi_1 \cap \eta_j \neq 0$, then $\xi_1 = \xi_1 \cap \eta_j = \eta_j$ since ξ_1 and η_j are simple ideal bundles.

It follows that $\xi_1 \cap \eta_j \neq 0$ for at most one η_j. On the other hand, if $\xi_1 \cap \eta_j = 0$ for all j then for each $x \in X$, $\xi_{1x} \cap \eta_{jx} = 0$. Therefore for each $x \in X$, $[\xi_{1x}, \eta_{jx}] \subset \xi_{1x} \cap \eta_{jx} = 0$. Since $\xi_x = \xi_{1x} \times \xi_{2x} \times \cdots \times \xi_{nx} = \eta_{1x} \times \cdots \times \eta_{kx}$, this implies that $[\xi_{1x}, \xi_{1x}] = 0$

contrary to the assumption that $[\xi_{1x},\xi_{1x}] \neq 0$. Hence there is a j such that $\xi_1 = \eta_j$. Similarly we have that every ξ_i coincides with one of the η_j and every η_j coincides with one of the ξ_i.

4. LIE BUNDLES AND LIE RINGS.

In this chapter we study Lie bundles in which all fibres need not be isomorphic as Lie algebras. Here we prove the full annalogue of Swan's theorem by giving a characterization of which Lie rings occur as the spaces of sections of Lie bundles. For the sake of completeness we prove some results in a nutshell though the proofs follow from the techniques adopted in [9].

DEFINITION 4.1. *A section of a Lie bundle* ξ *is a section of the underlying vector bundle* ξ. *The set of all sections of* ξ *is always denoted by* $\Gamma(\xi)$.

LEMMA 4.2. *Let* ξ *be a Lie bundle over* X : *then the set of all sections of* ξ, $\Gamma(\xi)$, *forms a Lie algebra over the ring* $C(X)$ *of all continuous* $\mathbb{R}$*-valued functions on* X.

Proof. Let $S_1, S_2 \in \Gamma(\xi)$. Then the map $X \xrightarrow{(S_1,S_2)} \xi \oplus \xi \xrightarrow{\theta} \xi$ defined by $x \longmapsto (S_1(x), S_2(x)) \longmapsto \theta_x(S_1(x), S_2(x))$ is continuous.

Therefore $[\ ] : \Gamma(\xi) \times \Gamma(\xi) \longrightarrow \Gamma(\xi)$

$$(S_1, S_2) \longrightarrow [S_1,S_2]$$

where $[S_1,S_2](x) = \theta_x(S_1(x), S_2(x))$ is continuous then $[S_1,S_2] \in \Gamma(\xi)$. Therefore $\Gamma(\xi)$ forms a Lie algebra over $C(X)$ since each ξ_x is a Lie algebra over $\mathbb{R}$.

LEMMA 4.3. Let $I_x = \{\alpha | \alpha(x) = 0,\ \alpha \in C(X)\}$ be the maximal ideal of $C \subset X$ attached to $x \in X$. Then the map $\theta : \Gamma(\xi)/I_x\Gamma(\xi) \to \xi_x$ given by $\theta(s + I_x\Gamma(\xi)) = s(x)$ is a Lie algebra isomorphism where $I_x\Gamma(\xi) = \{z | z = \sum_i \alpha_i s_i,\ \alpha_i \in I_x,\ s_i \in \Gamma(\xi)\}$ is an ideal in $\Gamma(\xi)$.

Proof. The map $\theta : \Gamma(\xi)/I_x\Gamma(\xi) \to \xi_x$ given by

$$\bar{S} = S + I_x\Gamma(\xi) \mapsto S(x)$$

is linear, and bijective [9, corollary 3, p. 267] and θ preserves the Lie product.

For,
$$\begin{aligned}\theta[\bar{S_1}, \bar{S_2}] &= \theta\overline{[S_1, S_2]} \\ &= [S_1, S_2](x) \\ &= [S_1(x),\ S_2(x)] \\ &= [\theta\bar{S_1},\ \theta\bar{S_2}].\end{aligned}$$

Thus the lemma is proved.

THEOREM 4.4. Let ξ, η be two Lie bundles over a normal base space X and $F : \Gamma(\xi) \to \Gamma(\eta)$ be a Lie ring isomorphism over $C(X)$. Then there is a unique Lie bundle isomorphism $f : \xi \to \eta$ such that $F = \Gamma(f)$.

Proof. The Lie ring isomorphism $F : \Gamma(\xi) \to \Gamma(\eta)$ induces an isomorphism $\bar{F}_x : \Gamma(\xi)/I_x\Gamma(\xi) \to \Gamma(\eta)/I_x\Gamma(\eta)$.

Since by lemma 4.3 there exists Lie algebra isomorphisms $\varphi : \Gamma(\xi)/I_x\Gamma(\xi) \to \xi_x$ and $\theta : \Gamma(\eta)/I_x\Gamma(\eta) \to \eta_x$, therefore

$$\xi_x \xrightarrow{\varphi^{-1}} \Gamma(\xi)/I_x\Gamma(\xi) \xrightarrow{\bar{F}_x} \Gamma(\eta)/I_x\Gamma(\eta) \xrightarrow{\theta} \eta_x$$ is a Lie algebra isomorphism.

Say $f_x = \theta \bar{F}_x \varphi^{-1} : \xi_x \rightarrow \eta_x$.

Now define a map $f : \xi \rightarrow \eta$ by $e \mapsto f_x(e)$ if $e \in \xi_x$.

Then f is well-defined and obviously it preserves the Lie product restricted to each fibre ξ_x. Thus f satisfies all the required conditions of the theorem [9, theorem 1, p. 267].

We will now prove the converse of the theorem.

THEOREM 4.5. *Let ξ, η be two Lie bundles over a base space X. If the Lie bundles ξ and η are isomorphic, then the Lie rings $\Gamma(\xi)$ and $\Gamma(\eta)$ are isomorphic over $C(X)$.*

Proof. Suppose $\theta : \xi \rightarrow \eta$ is a Lie bundle isomorphism. Then the map $\Gamma(\theta) : \Gamma(\xi) \rightarrow \Gamma(\eta)$ defined by $\Gamma(\theta)S = \theta \circ s$ is well-defined, bijective and linear [9, corollary 4, p. 267] and it preserves the Lie ring product, that is,

$$\Gamma(\theta)[S_1, S_2] = [\Gamma(\theta)S_1, \Gamma(\theta)S_2].$$

For,
$$\begin{aligned}\Gamma(\theta)[S_1, S_2](x) &= \theta[S_1, S_2](x) \\ &= \theta[S_1(x), S_2(x)] \\ &= [\theta S_1(x), \theta S_2(x)] \text{ since } \theta|_{\xi_x} \text{ is a Lie homomorphism} \\ &= [\theta S_1, \theta S_2](x) \text{ for all } x \in X.\end{aligned}$$

Thus the Lie rings $\Gamma(\xi)$ and $\Gamma(\eta)$ are isomorphic.

SWAN'S THEOREM 4.6. *Let X be a compact Hausdorff-space, then a Lie algebra P over $C(X)$ is isomorphic to a Lie algebra of the form $\Gamma(\xi)$ where*

ξ is a Lie bundle if and only if P is a finitely generated projective module over $C(X)$.

Proof. Suppose P is of the form $\Gamma(\xi)$ where ξ is a Lie bundle over X. Then P is finitely generated projective module [9, corollary 6, p. 268].

Suppose now that P is a finitely generated projective module. Then there exists a vector bundle ξ with $\Gamma(\xi)$ isomorphic to P as modules over $C(X)$ [9, Theorem 2, p. 268-269].

Now we will give ξ a Lie bundle structure so that the Lie ring $\Gamma(\xi)$ (Lie ring structure induced from ξ) is isomorphic to P as Lie rings.

Suppose $[\ ] : P \times P \longrightarrow P$

is the given Lie ring structure. Therefore

$$[\ ]_x : P/I_xP \times P/I_xP \longrightarrow P/I_xP$$

is a Lie algebra structure for each $x \in X$ and we know that

$\theta_x : P/I_xP \longrightarrow \xi_x$ is given by $S + I_xP \longrightarrow S(x)$ is a vector space isomorphism. Therefore we have

$$\begin{array}{ccc} P/I_xP \times P/I_xP & \longrightarrow & P/I_xP \\ \theta_x \times \theta_x \downarrow & & \downarrow \theta_x \\ \xi_x \times \xi_x & \longrightarrow & \xi_x \end{array}$$

$(\theta_x \times \theta_x) \circ \theta_x = \theta_x \circ [\ ]_x$.

Then obviously on each fibre ξ_x, $\theta_x : \xi_x \times \xi_x \longrightarrow \xi_x$ induces a Lie algebra structure over $\mathbb{R}$. The totality of these yield a map $\theta : \xi \times \xi \longrightarrow \xi$. We will now prove that θ is continuous. If $S_1, S_2 \in \Gamma(\xi)$, $\theta(S_1,S_2) = [S_1,S_2]$.

Indeed $\theta(S_1,S_2)(x) = \theta(S_1(x), S_2(x))$

$= \theta_x[\bar{S}_1,\bar{S}_2]_x$

$= [S_1,S_2]_x$

$= [S_1,S_2](x)$, for all $x \in X$.

To check that θ is continuous, let $\{S_i\}$ be a local base at $x \in X$ and $e = (e_1,e_2) \in \xi \oplus \xi$ such that $q(e) = p(e_1) = p(e_2)$ is near x. Then

$$(e_1,e_2) = (\sum_i a_i(e_1)\, S_i(p(e_1)),\ \sum_j b_j(e_2)\, S_j(p(e_2)))$$

where $a_i(e_1),\ b_j(e_2) \in C(X)$.

Therefore $\theta(e_1,e_2) = \theta(\sum_i a_i(e_1)\, S_i(p(e_1)),\ \sum_j b_j(e_2)\, S_j(p(e_2)))$

$$= \sum_i a_i(e_1) \sum_j b_j(e_2)\, \theta_x(S_i(p(e_1)), S_j(p(e_2))).$$

Since $\theta(S_i,S_j) = [S_i,S_j]$ is a continuous section of ξ. Therefore θ is continuous.

Thus ξ is a Lie bundle. Then by the very construction, the Lie ring $\Gamma(\xi)$ is isomorphic to the given Lie ring P.

Thus the theorem is proved.

(1) Every (locally trivial) Lie algebra bundle defines a Lie group bundle.

(2) If ξ is a Lie algebra bundle, and each ξ_x is a semi-simple Lie algebra, and X is connected, then the Lie algebras ξ_x are mutually isomorphic.

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